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Mix and Match

H-454 *Proposed by Larry Taylor, Rego Park, NY*
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Construct six distinct Fibonacci-Lucas identities such that

(a) Each identity consists of three terms;
(b) Each term is the product of two Fibonacci numbers;
(c) Each subscript is either a Fibonacci or a Lucas number.

Solutions by Stanley Rabinowitz, Westford, MA

Solution Set 1

Here are six identities that meet the requested conditions, although they are probably not what the proposer intended:

$$F_{F_2}F_{F_n} + F_{F_3}F_{F_n} = F_{F_4}F_{F_n}$$
$$F_{F_2}F_{L_n} + F_{F_3}F_{L_n} = F_{F_4}F_{L_n}$$
$$F_{F_3}F_{F_n} + F_{F_4}F_{F_n} = F_{L_3}F_{F_n}$$
$$F_{F_3}F_{L_n} + F_{F_4}F_{L_n} = F_{L_3}F_{L_n}$$
$$F_{F_4}F_{F_n} + F_{L_3}F_{F_n} = F_{F_5}F_{F_n}$$
$$F_{F_4}F_{L_n} + F_{L_3}F_{L_n} = F_{F_5}F_{L_n}$$

Solution Set 2

If numerical identities are acceptable, then we have the following identities (found by computer search):

$$F_2F_3 + F_4F_8 = F_5F_7$$
$$F_2F_8 + F_5F_{11} = F_3F_{13}$$
$$F_2F_{18} + F_5F_{11} = F_7F_{13}$$
$$F_3F_7 + F_4F_8 = F_2F_{11}$$
$$F_3F_{13} + F_8F_{18} = F_5F_{21}$$
$$F_5F_{21} + F_8F_{34} = F_{13}F_{29}$$
$$F_8F_{18} + F_{11}F_{21} = F_3F_{29}$$
$$F_{13}F_{29} + F_{18}F_{34} = F_5F_{47}$$

where all the subscripts are distinct in each example.

Solution Set 3

The numerical identities in Solution Set 2 suggest the following identities involving one parameter, i:

$$\begin{cases} F_{F_{i+4}}F_{L_{i+1}} + F_{F_{i+2}}F_{L_{i+2}} = F_{F_i}F_{L_{i+3}} & \text{if } i \text{ is not divisible by 3} \\ F_{F_{i+4}}F_{L_{i+1}} = F_{F_{i+2}}F_{L_{i+2}} + F_{F_i}F_{L_{i+3}} & \text{if } 3|i. \end{cases}$$

We will prove these by proving the equivalent single condition:

$$(1) \qquad F_{F_{i+4}}F_{L_{i+1}} - (-1)^{F_i}F_{F_{i+2}}F_{L_{i+2}} = F_{F_i}F_{L_{i+3}}.$$

To verify identity (1), we apply the known transformation

$$5F_m F_n = L_{m+n} - (-1)^n L_{m-n}$$

to get:

$$L_{F_{i+4}+L_{i+1}} - (-1)^{L_{i+1}} L_{F_{i+4}-L_{i+1}} - (-1)^{F_i} [L_{F_{i+2}+L_{i+2}} - (-1)^{L_{i+2}} L_{F_{i+2}-L_{i+2}}]$$
$$- L_{F_i+L_{i+3}} + (-1)^{L_{i+3}} L_{F_i - L_{i+3}} = 0.$$

This identity can be shown to be true because, of the six terms, it can be grouped into pairs of terms that cancel. Specifically,

(2) $$L_{F_{i+4}+L_{i+1}} = L_{F_i+L_{i+3}}$$

(3) $$(-1)^{L_{i+1}} L_{F_{i+4}-L_{i+1}} = (-1)^{F_i} (-1)^{L_{i+2}} L_{F_{i+2}-L_{i+2}}$$

(4) $$(-1)^{F_i} L_{F_{i+2}+L_{i+2}} = (-1)^{L_{i+3}} L_{F_i - L_{i+3}}$$

Equation (2) follows from the identity

$$F_{i+4} + L_{i+1} = F_i + L_{i+3},$$

which is straightforward to prove.

To prove equation (3), we use the fact that $L_{-n} = (-1)^n L_n$, so that

$$L_{F_{i+2}-L_{i+2}} = L_{-F_{i+2}+L_{i+2}}$$

since a simple parity argument shows that $F_{i+2} - L_{i+2}$ is always even. Then we note that $F_i + L_{i+2} \equiv L_{i+1} \pmod 2$, which also follows from a simple parity argument. Thus,

$$(-1)^{L_{i+1}} = (-1)^{F_i + L_{i+2}}$$

and we see that equation (3) is equivalent to

$$F_{i+4} - L_{i+1} = -F_{i+2} + L_{i+2},$$

which we again leave as a simple exercise for the reader.

For equation (4), we have similarly that $F_i \equiv L_{i+3} \pmod 2$, and hence equation (4) is equivalent to the easily proven

$$F_{i+2} + L_{i+2} = -F_i + L_{i+3},$$

where again we note that $F_i - L_{i+3}$ is always even.

Finally, we note a second identity analogous to (1):

(5) $$F_{F_{i+1}} F_{L_{i+1}} - (-1)^{F_i} F_{F_{i-1}} F_{F_{i+2}} = F_{F_i} F_{F_{i+3}}$$

whose proof is similar and is omitted.

Equations (1) and (5) appear to generate all the numerical examples I have found. If we let i have the forms $3k - 1$, $3k$, and $3k + 1$, we get the six identities:

$$F_{F_{3k+3}} F_{L_{3k}} + F_{F_{3k+1}} F_{L_{3k+1}} = F_{F_{3k-1}} F_{L_{3k+2}}$$
$$F_{F_{3k+4}} F_{L_{3k+1}} = F_{F_{3k+2}} F_{L_{3k+2}} + F_{F_{3k}} F_{L_{3k+3}}$$
$$F_{F_{3k+5}} F_{L_{3k+2}} + F_{F_{3k+3}} F_{L_{3k+3}} = F_{F_{3k+1}} F_{L_{3k+4}}$$
$$F_{F_{3k}} F_{L_{3k}} + F_{F_{3k-2}} F_{F_{3k+1}} = F_{F_{3k-1}} F_{F_{3k+2}}$$
$$F_{F_{3k+1}} F_{L_{3k+1}} = F_{F_{3k-1}} F_{F_{3k+2}} + F_{F_{3k}} F_{F_{3k+3}}$$
$$F_{F_{3k+2}} F_{L_{3k+2}} + F_{F_{3k}} F_{F_{3k+3}} = F_{F_{3k+1}} F_{F_{3k+4}}$$

which are probably the ones the proposer had in mind.